Global existence in critical spaces for incompressible viscoelastic fluids*

Ting Zhang,† Daoyuan Fang‡

Department of Mathematics, Zhejiang University, Hangzhou 310027, China

Abstract

We investigate local and global strong solutions for the incompressible viscoelastic system of Oldroyd–B type. We obtain the existence and uniqueness of a solution in a functional setting invariant by the scaling of the associated equations. More precisely, the initial velocity has the same critical regularity index as for the incompressible Navier–Stokes equations, and one more derivative is needed for the deformation tensor. We point out a smoothing effect on the velocity and a L^1−decay on the difference between the deformation tensor and the identity matrix. Our result implies that the deformation tensor F has the same regularity as the density of the compressible Navier–Stokes equations.

1 Introduction

In this paper, we consider the following system describing incompressible viscoelastic fluids.

$$\begin{cases} \nabla\cdot v=0, \quad x\in\mathbb{R}^N,\ N\geq 2, \\ v_t+v\cdot\nabla v+\nabla p=\mu\Delta v+\nabla\cdot\left[\frac{\partial W(F)}{\partial F}F^\top\right], \\ F_t+v\cdot\nabla F=\nabla vF, \\ F(0,x)=I+E_0(x),\ v(0,x)=v_0(x). \end{cases} \tag{1.1}$$

Here, v, p, $\mu>0$, F and $W(F)$ denote, respectively, the velocity field of materials, pressure, viscosity, deformation tensor and elastic energy functional. The third equation is simply the consequence of the chain law. It can also be regarded as the consistence condition of the flow trajectories obtained from the velocity field v and also of those obtained from the deformation tensor F ([7, 13, 17, 22, 24]). Moreover, on the right-hand side of the momentum equation, $\frac{\partial W(F)}{\partial F}$ is the Piola–Kirchhoff stress tensor and $\frac{\partial W(F)}{\partial F}F^\top$ is the Cauchy–Green tensor. The latter is the change variable (from Lagrangian to Eulerian coordinates) form of the former one ([17]). The above system is equivalent to the usual Oldroyd–B model for viscoelastic fluids in the case of infinite Weissenberg number ([16]). On the other hand, without the viscosity term, it represents exactly the incompressible elasticity in Eulerian coordinates. We refer to [2, 7, 16, 21, 23, 24] and their references for the detailed derivation and physical background of the above system.

Throughout this paper, we will use the notations of

$$(\nabla v)_{ij}=\frac{\partial v_i}{\partial x_j},\ (\nabla vF)_{ij}=(\nabla v)_{ik}F_{kj},\ (\nabla\cdot F)_i=\partial_j F_{ij},$$

and summation over repeated indices will always be understood.

For incompressible viscoelastic fluids, Lin et al.[22] proved the global existence of classical small solutions for the two–dimensional case with the initial data $v_0, E_0=F_0-I\in H^k(\mathbb{R}^2)$, $k\geq 2$, by introducing an auxiliary vector field to replace the transport variable F. Using the method in [14] for

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†zhangting79@hotmail.com

‡dyf@zju.edu.cn

the damped wave equation, they[22] also obtained the global existence of classical small solutions for the three–dimensional case with the initial data $v_0, E_0 \in H^k(\mathbb{R}^3)$, $k \geq 3$. Lei and Zhou[19] obtained the same results for the two–dimensional case with the initial data $v_0, E_0 \in H^s(\mathbb{R}^2)$, $s \geq 4$. via the incompressible limit working directly on the deformation tensor F. Then, Lei et al.[18] proved the global existence for two–dimensional small-strain viscoelasticity with $H^2(\mathbb{R}^2)$ initial data and without assumptions on the smallness of the rotational part of the initial deformation tensor. It is worth noticing that the global existence and uniqueness for the large solution of the two–dimensional problem is still open. Recently, by introducing an auxiliary function $w = \Delta v + \frac{1}{\mu}\nabla \cdot E$, Lei et al.[17] obtained a weak dissipation on the deformation F and the global existence of classical small solutions to N–dimensional system with the initial data $v_0, E_0 \in H^2(\mathbb{R}^N)$ and $N = 2, 3$. All these results need that the initial data v_0 and E_0 have the same regularity, and the regularity index of the initial velocity v_0 is bigger than the critical regularity index for the classical incompressible Navier–Stokes equations. In this paper, using Danchin's method to study the compressible Navier–Stokes system, we are concerned with the existence and uniqueness of a solution for the initial data in a functional space with minimal regularity order. Although the equation $(1.1)_3$ for the deformation tensor F is identical to the equation for the vorticity $\omega = \nabla \times v$ of the Euler equations

$$\partial_t \omega + v \cdot \nabla \omega = \nabla v \omega,$$

our result implies that the deformation tensor F has the same regularity as the density of the compressible Navier–Stokes equations.

At this stage, we will use scaling considerations for (1.1) to guess which spaces may be critical. We observe that (1.1) is invariant by the transformation

$$(v_0(x), F_0(x)) \to (lv_0(lx), F_0(lx)),$$

$$(v(t,x), F(t,x), P(t,x)) \to (lv(l^2t, lx), F(l^2t, lx), l^2P(l^2t, lx)),$$

up to a change of the elastic energy functional W into l^2W.

Definition 1.1. A functional space $E \subset (\mathcal{S}'(\mathbb{R}^N))^N \times (\mathcal{S}'(\mathbb{R}^N)^{N\times N}$ is called a critical space if the associated norm is invariant under the transformation $(v(x), F(x)) \to (lv(lx), F(lx))$ (up to a constant independent of l).

Obviously $(\dot{H}^{\frac{N}{2}-1})^N \times (\dot{H}^{\frac{N}{2}})^{N\times N}$ is a critical space for the initial data. The space $\dot{H}^{\frac{N}{2}}$ however is not included in L^∞, we cannot expect to get L^∞ control on the deformation tensor, when we choose $(F_0 - I) \in (\dot{H}^{\frac{N}{2}})^{N\times N}$. Moreover, the product between functions does not extend continuously from $\dot{H}^{\frac{N}{2}-1} \times \dot{H}^{\frac{N}{2}}$ to $\dot{H}^{\frac{N}{2}-1}$, so that we will run into difficulties when estimating the nonlinear terms. Similar to the compressible Navier–Stokes system ([8]), we could use homogeneous Besov spaces $B^s := \dot{B}^s_{2,1}(\mathbb{R}^N)$ (refer to Sect. 3 for the definition of such spaces) with the same derivative index. Now, $B^{\frac{N}{2}}$ is an algebra embedded in L^∞. This allows us to control the deformation tensor from above without requiring more regularity on derivatives of F. Moreover, the product is continuous from $B^{\frac{N}{2}-\alpha} \times B^{\frac{N}{2}}$ to $B^{\frac{N}{2}-\alpha}$ for $0 \leq \alpha < N$, and $(B^{\frac{N}{2}-1})^N \times (B^{\frac{N}{2}})^{N\times N}$ is a critical space.

In this paper, we assume that $E_0 := F_0 - I$ and v_0 satisfy the following constraints:

$$\nabla \cdot v_0 = 0, \ \det(I + E_0) = 1, \ \nabla \cdot E_0^\top = 0, \tag{1.2}$$

and

$$\partial_m E_{0ij} - \partial_j E_{0im} = E_{0lj}\partial_l E_{0im} - E_{0lm}\partial_l E_{0ij}. \tag{1.3}$$

The first three of these expressions are just the consequences of the incompressibility condition and the last one can be understood as the consistency condition for changing variables between the Lagrangian and Eulerian coordinates [17].

For simplicity, we only consider the case of Hookean elastic materials: $W(F) = \frac{1}{2}|F|^2 = \frac{1}{2}\mathrm{tr}(FF^\top)$. Define the usual strain tensor by the form

$$E = F - I. \tag{1.4}$$

Then, the system (1.1) is

$$\begin{cases} \nabla \cdot v = 0, \quad x \in \mathbb{R}^N, \ N \geq 2, \\ v_{it} + v \cdot \nabla v_i + \partial_i p = \mu \Delta v_i + E_{jk} \partial_j E_{ik} + \partial_j E_{ij}, \\ E_t + v \cdot \nabla E = \nabla v E + \nabla v, \\ (v, E)(0, x) = (v_0, E_0)(x). \end{cases} \tag{1.5}$$

Let us now state our main results. Define the following functional space:

$$U_T^s = \left(L^1([0,T]; B^{s+1}) \cap C([0,T]; B^{s-1}) \right)^N \times \left(C([0,T]; B^s) \right)^{N \times N},$$

$$V^s = \left(L^1(\mathbb{R}^+; B^{s+1}) \cap C(\mathbb{R}^+; B^{s-1}) \right)^N \times \left(L^2(\mathbb{R}^+; B^s) \cap C(\mathbb{R}^+; B^s \cap B^{s-1}) \right)^{N \times N}.$$

Theorem 1.1 (Local result)**.** *Suppose that the initial data satisfies the incompressible constraints (1.2) and $E_0 \in B^{\frac{N}{2}}$, $v_0 \in B^{\frac{N}{2}-1}$. Then the following results hold true:*

1) *There exist $T > 0$ and a unique local solution for system (1.5) that satisfies*

$$(v, E) \in \left(L^1([0,T]; B^{\frac{N}{2}+1}) \cap C([0,T]; B^{\frac{N}{2}-1}) \right)^N \times \left(C([0,T]; B^{\frac{N}{2}}) \right)^{N \times N},$$

$$\|(v, E)\|_{U_T^{\frac{N}{2}}} \leq C(\|E_0\|_{B^{\frac{N}{2}}} + \|v_0\|_{B^{\frac{N}{2}-1}}), \tag{1.6}$$

and

$$\nabla \cdot v = 0, \ \det(I + E) = 1, \ \nabla \cdot E^\top = 0. \tag{1.7}$$

2) *Moreover, if $E_0 \in B^s$ and $u_0 \in B^{s-1}$, $s \in (\frac{N}{2}, \frac{N}{2}+1)$, then*

$$\|(v, E)\|_{U_T^s} \leq C(\|E_0\|_{B^s} + \|v_0\|_{B^{s-1}}). \tag{1.8}$$

Theorem 1.2 (Global result)**.** *Suppose that the initial data satisfies the incompressible constraints (1.2)–(1.3), $E_0 \in B^{\frac{N}{2}} \cap B^{\frac{N}{2}-1}$, $v_0 \in B^{\frac{N}{2}-1}$ and*

$$\|E_0\|_{B^{\frac{N}{2}} \cap B^{\frac{N}{2}-1}} + \|v_0\|_{B^{\frac{N}{2}-1}} \leq \lambda, \tag{1.9}$$

where λ is a small positive constant. Then the following results hold true:

1) *There exists a unique global solution for system (1.5) that satisfies*

$$v \in \left(L^1(\mathbb{R}^+; B^{\frac{N}{2}+1}) \cap C(\mathbb{R}^+; B^{\frac{N}{2}-1}) \right)^N,$$

$$E \in \left(L^2(\mathbb{R}^+; B^{\frac{N}{2}}) \cap C(\mathbb{R}^+; B^{\frac{N}{2}} \cap B^{\frac{N}{2}-1}) \right)^{N \times N},$$

$$\|(v, E)\|_{V^{\frac{N}{2}}} \leq M(\|E_0\|_{B^{\frac{N}{2}} \cap B^{\frac{N}{2}-1}} + \|v_0\|_{B^{\frac{N}{2}-1}}). \tag{1.10}$$

2) *Moreover, if $E_0 \in B^s$ and $u_0 \in B^{s-1}$, $s \in (\frac{N}{2}, \frac{N}{2}+1)$, then*

$$\|(v, E)\|_{V^s} \leq C(\|E_0\|_{B^s} + \|v_0\|_{B^{s-1}}). \tag{1.11}$$

Remark 1.1. The L^2-decay in time for E is a key point in the proof of the global existence. We shall also get a L^1-decay in a space a trifle larger than $B^{\frac{N}{2}}$ (see Theorem 6.1 below).

Remark 1.2. Theorem 1.2 implies that the deformation tensor F has similar property as the density of the compressible Navier–Stokes system in [8]. And we think that the incompressible viscoelastic system is similar to the compressible Navier–Stokes system.

Remark 1.3. Similar to the compressible Navier–Stokes system in [8], the initial data do not really belong to a critical space in the sense of Definition 1.1. We indeed made the additional assumption $E_0 \in B^{\frac{N}{2}-1}$ (which however involves only low frequencies and does not change the required local regularity for E_0). On the other hand, our scaling considerations do not take care of the Cauchy–Green tensor term. A careful study of the linearized system (see Proposition 4.3 below) besides indicates that such an assumption may be unavoidable.

Remark 1.4. Considering the general viscoelastic model (1.1), if the strain energy function satisfies the strong Legendre–Hadamard ellipticity condition

$$\frac{\partial^2 W(I)}{\partial F_{il}\partial F_{jm}} = (\alpha^2 - 2\beta^2)\delta_{il}\delta jm + \beta^2(\delta_{im}\delta_{jl} + \delta_{ij}\delta_{lm}), \text{ with } \alpha > \beta > 0, \tag{1.12}$$

and the reference configuration stress–free condition

$$\frac{\partial W(I)}{\partial F} = 0, \tag{1.13}$$

then we can obtain the same results as that in Theorems 1.1–1.2.

In this paper, we introduce the following function:

$$c = \Lambda^{-1}\nabla \cdot E,$$

where $\Lambda^s f = \mathcal{F}^{-1}(|\xi|^s \hat{f})$. Then, the system (1.5) reads

$$\begin{cases} \nabla \cdot v = \nabla \cdot c = 0, \quad x \in \mathbb{R}^N, \ N \geq 2, \\ v_{it} + v \cdot \nabla v_i + \partial_i p = \mu\Delta v_i + E_{jk}\partial_j E_{ik} + \Lambda c_i, \\ c_t + v \cdot \nabla c + [\Lambda^{-1}\nabla\cdot, v\cdot]\nabla E = \Lambda^{-1}\nabla \cdot (\nabla v E) - \Lambda v, \\ \Delta E_{ij} = \Lambda\partial_j c_i + \partial_k(\partial_k E_{ij} - \partial_j E_{ik}), \\ (v, c)(0, x) = (v_0, \Lambda^{-1}\nabla \cdot E_0)(x). \end{cases} \tag{1.14}$$

So, we need to study the following mixed parabolic–hyperbolic linear system with a convection term:

$$\begin{cases} \nabla \cdot v = \nabla \cdot c = 0, \quad x \in \mathbb{R}^N, \ N \geq 2, \\ v_{it} + u \cdot \nabla v_i + \partial_i p - \mu\Delta v_i - \Lambda c_i = G, \\ c_t + u \cdot \nabla c + \Lambda v = L, \\ (v, c)(0, x) = (v_0, c_0)(x), \end{cases} \tag{1.15}$$

where $\mathrm{div} u = 0$ and $c_0 = \Lambda^{-1}\nabla \cdot E_0$. This system is similar to the system

$$\begin{cases} c_t + v \cdot \nabla c + \Lambda d = F, \\ d_t + v \cdot \nabla d - \bar{\mu}\Delta d - \Lambda c = G, \end{cases} \tag{1.16}$$

in Danchin's paper ([8]). Using the method studying the compressible Navier–Stokes system in [8] (Proposition 2.3), we can obtain the Proposition 4.3 and L^1-decay on E.

When we finished this paper, we noticed that some similar results were also obtained independently in [25], where J.Z. Qian introduced the function $d_{ij} = -\Lambda^{-1}\nabla_j v_i$, and considered the mixed parabolic–hyperbolic system of (E, d).

As for the related studies on the existence of solutions to nonlinear elastic systems, there are works by Sideris[26] and Agemi[1] on the global existence of classical small solutions to three–dimensional compressible elasticity, under the assumption that the nonlinear terms satisfy the null conditions. The global existence for three–dimensional incompressible elasticity was then proved via the incompressible limit method ([27]) and by a different method ([28]). It is worth noticing that the global existence and uniqueness for the corresponding two–dimensional problem is still open.

Now, let us recall some classical results for the following incompressible Navier–Stokes equations.

$$\begin{cases} v_t + v \cdot \nabla v + \nabla p = \mu\Delta v, \\ \nabla \cdot v = 0, \\ v(0, x) = v_0(x). \end{cases} \tag{1.17}$$

In 1934, J. Leray proved the existence of global weak solutions for (1.17) with divergence–free $u_0 \in L^2$ (see [20]). Then, H. Fujita and T. Kato[12] obtain the uniqueness with $u_0 \in \dot{H}^{\frac{N}{2}-1}$. The index $s = N/2 - 1$ is critical for (1.17) with initial data in $\dot{H}^s$: this is the lowest index for which uniqueness has been proved (in the framework of Sobolev spaces). This fact is closely linked to the concept of scaling invariant space. Let us precise what we mean. For all $l > 0$, system (1.17) is obviously invariant by the transformation

$$u(t,x) \to u_l(t,x) := lu(l^2 t, lx), u_0(x) \to u_{0l}(x) := lu_0(lx),$$

and a straightforward computation shows that $\|u_0\|_{\dot{H}^{\frac{N}{2}-1}} = \|u_{0l}\|_{\dot{H}^{\frac{N}{2}-1}}$. This idea of using a functional setting invariant by the scaling of (1.17) is now classical and originated many works. Refer to [3, 5, 6, 15] for a recent panorama.

The rest of this paper is organized as follows. In Section 2, we state three lemmas describing the intrinsic properties of viscoelastic system. In Section 3, we present the functional tool box: Littlewood–Paley decomposition, product laws in Sobolev and hybrid Besov spaces. The next section is devoted to the study of some linear models associated to (1.5). In Section 5, we will study the local well-posedness for (1.5). At last, we will prove Theorem 1.2 in Section 6.

2 Basic mechanics of viscoelasticity

Using the similar arguments as that in the proof of Lemmas 1–3 in [17], we can easily obtain the following three lemmas.

Lemma 2.1. *Assume that* $\det(I + E_0) = 1$ *is satisfied and* (v, F) *is the solution of system (1.5). Then the following is always true:*

$$\det(I + E) = 1, \tag{2.1}$$

for all time $t \geq 0$*, where the usual strain tensor* $E = F - I$.

Lemma 2.2. *Assume that* $\nabla \cdot E_0^\top = 0$ *is satisfied, then the solution* (v, F) *of system (1.5) satisfies the following identities:*

$$\nabla \cdot F^\top = 0, \quad \textit{and} \quad \nabla \cdot E^\top = 0, \tag{2.2}$$

for all time $t \geq 0$.

Lemma 2.3. *Assume that (1.3) is satisfied and* (v, F) *is the solution of system (1.5). Then the following is always true:*

$$\partial_m E_{ij} - \partial_j E_{im} = E_{lj} \partial_l E_{im} - E_{lm} \partial_l E_{ij}, \tag{2.3}$$

for all time $t \geq 0$.

3 Littlewood–Paley theory and Besov spaces

The proof of most of the results presented in this paper requires a dyadic decomposition of Fourier variable (*Littlewood–Paley composition*). Let us briefly explain how it may be built in the case $x \in \mathbb{R}^N$, $N \geq 2$, (see [8, 9, 10]).

Let $\mathcal{S}(\mathbb{R}^N)$ be the Schwarz class. $\varphi(\xi)$ is a smooth function valued in [0,1] such that

$$\mathrm{supp}\varphi \subset \{\frac{3}{4} \leq |\xi| \leq \frac{8}{3}\} \ \text{ and } \sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1, \ |\xi| \neq 0.$$

Let $h(x) = (\mathcal{F}^{-1}\varphi)(x)$. For $f \in \mathcal{S}'$ (denote the set of temperate distributes, which is the dual one of $\mathcal{S}$), we can define the homogeneous dyadic blocks as follows:

$$\Delta_q f(x) := \varphi(2^{-q}D) f(x) = 2^{Nq} \int_{\mathbb{R}^N} h(2^q y) f(x - y) dy, \ \text{if } q \in \mathbb{Z},$$

where $\mathcal{F}^{-1}$ represents the inverse Fourier transform. Define the low frequency cut-off by

$$S_q f(x) := \sum_{p\leq q-1} \Delta_p f(x) = \chi(2^{-q}D)f(x).$$

The Littlewood–Paley decomposition has nice properties of quasi-orthogonality,

$$\Delta_p\Delta_q f_1 \equiv 0, \text{ if } |p-q|\geq 2,$$

and

$$\Delta_q(S_{p-1}f_1\Delta_p f_2) \equiv 0, \text{ if } |p-q|\geq 5.$$

The Besov space can be characterized in virtue of the Littlewood–Paley decomposition.

Definition 3.1. Let $1\leq p\leq\infty$ and $s\in\mathbb{R}$. For $1\leq r\leq\infty$, the Besov spaces $\dot{B}^s_{p,r}(\mathbb{R}^N)$, $N\geq 2$, are defined by

$$f\in\dot{B}^s_{p,r}(\mathbb{R}^N)\Leftrightarrow \left\|2^{qs}\|\Delta_q f\|_{L^p(\mathbb{R}^N)}\right\|_{l^r_q}<\infty$$

and $B^s(\mathbb{R}^N)=\dot{B}^s_{2,1}(\mathbb{R}^N)$.

The definition of $\dot{B}^s_{p,r}(\mathbb{R}^N)$ does not depend on the choice of the Littlewood–Paley decomposition. Let us recall some classical estimates in Sobolev spaces for the product of two functions ([11]).

Proposition 3.1. *Let $1\leq r,p,p_1,p_2\leq\infty$. Then following inequalities hold true:*

$$\|uv\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{\dot{B}^s_{p,r}}, \quad \text{if} \ \ s>0, \tag{3.1}$$

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{N}{p}}_{p,r}} \lesssim \|u\|_{\dot{B}^{s_1}_{p,r}}\|v\|_{\dot{B}^{s_2}_{p,\infty}}, \quad \text{if} \ \ s_1,s_2<\frac{N}{p} \ \ \text{and} \ \ s_1+s_2>0, \tag{3.2}$$

$$\|uv\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{\dot{B}^s_{p,r}}\|v\|_{\dot{B}^{\frac{N}{p}}_{p,\infty}\cap L^\infty}, \quad \text{if} \ \ |s|<\frac{N}{p}, \tag{3.3}$$

$$\|uv\|_{\dot{B}^{-\frac{N}{p}}_{p,\infty}} \lesssim \|u\|_{\dot{B}^s_{p,1}}\|v\|_{\dot{B}^{-s}_{p,\infty}}, \quad \text{if} \ \ s\in(-\frac{N}{p},\frac{N}{p}], \ p\geq 2. \tag{3.4}$$

We give the following definition of hybrid Besov norms as that in [8].

Definition 3.2. For $\mu>0$, $r\in[1,+\infty]$ and $s\in\mathbb{R}$, we denote

$$\|u\|_{\widetilde{B}^{s,r}_\mu} := \sum_{q\in\mathbb{Z}} 2^{qs}\max\{\mu,2^{-q}\}^{1-\frac{2}{r}}\|\Delta_q u\|_{L^2}.$$

Let us remark that $\|f\|_{\widetilde{B}^{s,\infty}_\mu}\approx\|f\|_{B^s\cap B^{s-1}}$ and $\|f\|_{\widetilde{B}^{s,2}_\mu}=\|f\|_{B^s}$. Let us recall some estimates in hybrid Besov spaces for the product of two functions.

Proposition 3.2 ([8], Proposition 5.3)**.** *Let $r\in[1,\infty]$ and $s,t\in\mathbb{R}$. There exists some constant C such that*

$$\|T_u v\|_{\tilde{B}^{s+t-\frac{N}{2},r}_\mu} \leq C\|u\|_{\tilde{B}^{s,r}_\mu}\|v\|_{B^t}, \ \text{if} \ \ s\leq\min\{1-\frac{2}{r}+\frac{N}{2},\frac{N}{2}\},$$

$$\|T_u v\|_{\tilde{B}^{s+t-\frac{N}{2},r}_\mu} \leq C\|u\|_{B^s}\|v\|_{\tilde{B}^{t,r}_\mu}, \ \text{if} \ \ s\leq\frac{N}{2},$$

$$\|R(u,v)\|_{\tilde{B}^{s+t-\frac{N}{2},r}_\mu} \leq C\|u\|_{\tilde{B}^{s,r}_\mu}\|v\|_{B^t}, \ \text{if} \ \ s+t>\max\{0,1-\frac{2}{r}\},$$

where

$$T_f g = \sum_{p\leq q-2}\Delta_p f\Delta_q g = \sum_q S_{q-1}f\Delta_q g$$

and

$$R(f,g) = \sum_q \Delta_q f\widetilde{\Delta}_q g \ \textit{with} \ \widetilde{\Delta}_q := \Delta_{q-1}+\Delta_q+\Delta_{q+1}.$$

Lemma 3.1.

$$\big\|[\Lambda^{-1}\nabla, u]\nabla v\big\|_{\tilde{B}_\mu^{\frac{N}{2},\infty}} \le C\|\nabla u\|_{B^{\frac{N}{2}}}\|v\|_{\tilde{B}_\mu^{\frac{N}{2},\infty}}. \tag{3.5}$$

Proof. Since

$$\begin{aligned}
\big|\mathcal{F}([\Lambda^{-1}\partial_i, u]\partial_j v)\big| &= \left|\int_{\mathbb{R}^N} \frac{\xi_i}{|\xi|}\hat{u}(\xi-\eta)\eta_j\hat{v}(\eta) - \hat{u}(\xi-\eta)\frac{\eta_i}{|\eta|}\eta_j\hat{v}(\eta)d\eta\right| \\
&= \left|\int_{\mathbb{R}^N} \frac{\xi_i|\eta| - \eta_i|\xi|}{|\xi||\eta|}\eta_j\hat{u}(\xi-\eta)\hat{v}(\eta)d\eta\right| \\
&= \left|\int_{\mathbb{R}^N} \frac{\xi_i(|\eta|-|\xi|) + (\xi_i-\eta_i)|\xi|}{|\xi||\eta|}\eta_j\hat{u}(\xi-\eta)\hat{v}(\eta)d\eta\right| \\
&\le \int_{\mathbb{R}^N} |\eta-\xi||\hat{u}(\xi-\eta)||\hat{v}(\eta)|d\eta \\
&= \int_{\mathbb{R}^N} \widehat{(\nabla u)^*}(\xi-\eta)\widehat{v^*}(\eta)d\eta = \mathcal{F}((\nabla u)^* v^*),
\end{aligned}$$

where $f^* = \mathcal{F}^{-1}(|\hat{f}|)$, we have

$$\begin{aligned}
\|\Delta_p([\Lambda^{-1}\partial_i, u]\partial_j v))\|_{L^2} &= C\|\varphi(2^{-p}\xi)\mathcal{F}([\Lambda^{-1}\partial_i, u]\partial_j v))\|_{L^2_\xi} \\
&\le C\|\varphi(2^{-p}\xi)\mathcal{F}((\nabla u)^* v^*)\|_{L^2_\xi} \\
&= C\|\Delta_p((\nabla u)^* v^*)\|_{L^2}
\end{aligned}$$

and from Proposition 3.2,

$$\begin{aligned}
\big\|[\Lambda^{-1}\nabla, u]\nabla v\big\|_{\tilde{B}_\mu^{\frac{N}{2},\infty}} &\le C\,\|(\nabla u)^* v^*\|_{\tilde{B}_\mu^{\frac{N}{2},\infty}} \\
&\le C\|(\nabla u)^*\|_{B^{\frac{N}{2}}}\|v^*\|_{\tilde{B}_\mu^{\frac{N}{2},\infty}} \\
&\le C\|\nabla u\|_{B^{\frac{N}{2}}}\|v\|_{\tilde{B}_\mu^{\frac{N}{2},\infty}},
\end{aligned}$$

where we use the face that $\|\Delta_p f^*\|_{L^2} = \|\varphi(2^{-p}\xi)|\hat{f}|\|_{L^2_\xi} = \|\varphi(2^{-p}\xi)\hat{f}(\xi)\|_{L^2_\xi} = \|\Delta_p f\|_{L^2}$. ☐

Then, we give the definition of the Chemin-Lerner type spaces.

Definition 3.3. Let $s \in \mathbb{R}$, $(r, \lambda, p) \in [1, +\infty]^3$ and $T \in]0, +\infty]$. We define $\widetilde{L}^\lambda_T(\dot{B}^s_{p\,r})$ as the completion of $C([0,T], \mathcal{S})$ by the norm

$$\|f\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})} := \Big(\sum_{q\in\mathbb{Z}} 2^{qrs}\Big(\int_0^T \|\Delta_q\, f(t)\|^\lambda_{L^p}\, dt\Big)^{\frac{r}{\lambda}}\Big)^{\frac{1}{r}} < \infty.$$

with the usual change if $\lambda = \infty$, $r = \infty$.

4 A linear model with convection

Using the similar argument as that in the proof of Proposition A.1 in [9], we can obtain the following proposition.

Proposition 4.1. *Let $(p, r) \in [1, +\infty]^2$ and $s \in (-\frac{N}{p}, 1 + \frac{N}{p})$, and v be a solenoidal vector field such that $\nabla v \in L^1(0, T; \dot{B}^{\frac{N}{p}}_{p,r} \cap L^\infty)$. Suppose that $f_0 \in \dot{B}^s_{p,r}$, $g \in L^1(0, T; \dot{B}^s_{p,r})$ and that $f \in L^\infty(0, T; \dot{B}^s_{p,r}) \cap C([0, T]; \mathcal{S}')$ solves*

$$\begin{cases} \partial_t f + \nabla\cdot(vf) = g, \\ f|_{t=0} = f_0. \end{cases}$$

Then there exists a constant C depending only on s, p, r and N, such that the following inequality holds true for $t \in [0,T]$:

$$\|f\|_{\widetilde{L}^\infty_t(\dot{B}^s_{p,r})} \le e^{C\widetilde{V}(t)} \left(\|f_0\|_{\dot{B}^s_{p,r}} + \int_0^t e^{-C\widetilde{V}(t)} \|g(t)\|_{\dot{B}^s_{p,r}} dt \right), \tag{4.1}$$

with $\widetilde{V}(t) = \int_0^t \|\nabla v\|_{\dot{B}^{\frac{N}{p}}_{p,r} \cap L^\infty} d\tau$. Moreover, if $r < \infty$, then $f \in C([0,T]; \dot{B}^{\frac{N}{p}}_{p,r})$.

Proposition 4.2 ([10], Prop. 3.2)**.** *Let $s \in (-\frac{N}{2}, 2 + \frac{N}{2})$, $r \in [1,\infty]$, u_0 be a divergence-free vector field with coefficients in $\dot{B}^{s-1}_{2,r}$ and $f \in \widetilde{L}^1_T(\dot{B}^{s-1}_{2,r})$. Let u, v be two divergence-free time dependent vector fields such that $\nabla v \in L^1(0,T; \dot{B}^{\frac{N}{2}}_{2,r} \cap L^\infty)$ and $u \in C([0,T); \dot{B}^{s-1}_{2,r}) \cap \widetilde{L}^1_T(\dot{B}^{s+1}_{2,r})$. Assume in addition that*

$$\begin{cases} u_t + v \cdot \nabla u - \mu \Delta u + \nabla \Pi = f, \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{4.2}$$

is fulfilled for some distribution Π. Then, there exists $C = C(s,r,N)$ such that the following estimate holds true for $t \in [0,T]$:

$$\begin{aligned} & \|u\|_{\widetilde{L}^\infty_t(\dot{B}^{s-1}_{2,r})} + \mu \|u\|_{\widetilde{L}^1_t(\dot{B}^{s+1}_{2,r})} + \|\nabla \Pi\|_{\widetilde{L}^1_t(\dot{B}^{s-1}_{2,r})} \\ \le\ & e^{C\widetilde{V}(t)} \left(\|u_0\|_{\dot{B}^{s-1}_{2,r}} + C\|f\|_{\widetilde{L}^1_T(\dot{B}^{s-1}_{2,r})} \right), \end{aligned} \tag{4.3}$$

where $\widetilde{V}(t) = \int_0^t \|\nabla v\|_{\dot{B}^{\frac{N}{2}}_{2,r} \cap L^\infty} d\tau$.

After the change of function

$$c = \Lambda^{-1} \nabla \cdot E,$$

the system (1.5) reads (1.14). At first, we will study the following mixed linear system

$$\begin{cases} \nabla \cdot v = \nabla \cdot c = 0, \quad x \in \mathbb{R}^N,\ N \ge 2, \\ v_{it} + u \cdot \nabla v_i + \partial_i p - \mu \Delta v_i - \Lambda c_i = G, \\ c_t + u \cdot \nabla c + \Lambda v = L, \\ (v,c)(0,x) = (v_0, c_0)(x), \end{cases} \tag{4.4}$$

where $\mathrm{div} u = 0$ and $c_0 = \Lambda^{-1} \nabla \cdot E_0$. Using the similar arguments as that in [8] (Proposition 2.3), we can obtain the following proposition and omit the details. The main different is that there is a pressure term ∇p in $(4.4)_2$. Using that fact that $\nabla \cdot v = \nabla \cdot c = 0$, we have $\int v \cdot \nabla p = \int c \cdot \nabla p = 0$, and obtain the following proposition.

Proposition 4.3. *Let (v,c) be a solution of (4.4) on $[0,T)$, $1 - \frac{N}{2} < \rho \le 1 + \frac{N}{2}$ and $\widetilde{U}(t) = \int_0^t \|u(\tau)\|_{B^{\frac{N}{2}+1}} d\tau$. The following estimate holds on $[0,T)$:*

$$\begin{aligned} \|(v,c)\|_{X^\rho_T} \le C e^{C\widetilde{U}(T)} \big(\quad & \|c_0\|_{\tilde{B}^{\rho,\infty}_\mu} + \|v_0\|_{B^{\rho-1}} \\ & + \int_0^t e^{-C\widetilde{U}(s)} \left(\|L(s)\|_{\tilde{B}^{\rho,\infty}_\mu} + \|G(s)\|_{B^{\rho-1}} \right) ds \big), \end{aligned}$$

where C depends only on N and ρ,

$$\begin{aligned} X^s_T = \big\{ (v,c) \in \quad & \left(L^1(0,T; B^{s+1}) \cap C(0,T; B^{s-1}) \right)^N \\ & \times \left(L^1(0,T; \tilde{B}^{s,1}_\mu) \cap C(0,T; \tilde{B}^{s,\infty}_\mu) \right)^N \big\}, \end{aligned}$$

and $\|(v,c)\|_{X^s_T} = \|v\|_{L^\infty_T(B^{s-1})} + \mu \|v\|_{L^1_T(B^{s+1})} + \|c\|_{L^\infty_T(\tilde{B}^{s,\infty}_\mu)} + \mu \|c\|_{L^1_T(\tilde{B}^{s,1}_\mu)}$.

5 Local results

5.1 Local existence

Let $\bar{v}$ be the solution of the linear heat equations,

$$\begin{cases} \partial_t \bar{v} - \mu\Delta\bar{v} = 0, \\ \bar{v}(0,x) = v_0(x). \end{cases} \tag{5.1}$$

It is easy to obtain that $\bar{v} \in C([0,T]; B^{\frac{N}{2}-1})$ and

$$\|\bar{v}\|_{L^r_T(B^{\frac{N}{2}-1+\frac{2}{r}})} \leq C\|v_0\|_{B^{\frac{N}{2}-1}} \; r \in [1,\infty]. \tag{5.2}$$

Then, we can choose $T_1 \in (0,1)$ such that

$$\|\bar{v}\|_{L^1_{T_1}(B^{\frac{N}{2}+1})} \leq \lambda_1, \tag{5.3}$$

where λ_1 is chosen in (5.7) and (5.9).

Let $u = v - \bar{v}$. We will obtain the local existence of the solution to the following system

$$\begin{cases} \nabla\cdot u = 0, \quad x \in \mathbb{R}^N, \; N \geq 2, \\ u_{it} + u\cdot\nabla u_i + u\cdot\nabla\bar{v}_i + \bar{v}\cdot\nabla u_i + \bar{v}\cdot\nabla\bar{v}_i + \partial_i p = \mu\Delta u_i + E_{jk}\partial_j E_{ik} + \partial_j E_{ij}, \\ E_t + u\cdot\nabla E + \bar{v}\cdot\nabla E = \nabla u E + \nabla\bar{v}E + \nabla u + \nabla\bar{v}, \\ (u,E)(0,x) = (0,E_0)(x). \end{cases} \tag{5.4}$$

Now, we use an iterative method to build approximate solutions (u^n, E^n) of (5.4) which are solutions of linear system. Set the first term (E^0, u^0) to $(0,0)$. Then define $\{(u^n,E^n)\}_{n\in\mathbb{N}}$ by induction. Choose (E^{n+1}, v^{n+1}) as the solution of the following linear system

$$\begin{cases} \nabla\cdot u^{n+1} = 0, \quad x \in \mathbb{R}^N, \; N \geq 2, \\ (u^{n+1})_{it} + (u^n+\bar{v})\cdot\nabla(u^{n+1})_i + \partial_i p^{n+1} - \mu\Delta(u^{n+1})_i \\ \qquad\qquad = -(u^n+\bar{v})\cdot\nabla\bar{v}_i + E^{n+1}_{jk}\partial_j E^{n+1}_{ik} + \partial_j E^{n+1}_{ij}, \\ E^{n+1}_t + (u^n+\bar{v})\cdot\nabla E^{n+1} = \nabla u^n E^n + \nabla\bar{v}E^n + \nabla u^n + \nabla\bar{v}, \\ (u^{n+1},E)(0,x) = (0,E_0)(x). \end{cases} \tag{5.5}$$

The existence of the third equation can be obtained by the classical result of the transport equations. About the second equation's existence result, we can use the Friedrichs mollifiers. Here, we omit the details.

We are going to prove that, if $T \in (0,1)$ is small enough, the following bound holds for all $n \in \mathbb{N}$,

$$\|E^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}})} \leq 6\|E_0\|_{B^{\frac{N}{2}}}, \; \|u^n\|_{\widetilde{L}^\infty_T(B^{s-1})} + \|u^n\|_{L^1_T(B^{s+1})} \leq \lambda_1. \tag{P_n}$$

Suppose that (P_n) is satisfied and let us prove that (P_{n+1}) is also true.

From Propositions 3.1 and 4.1, we get

$$\begin{aligned} & \|E^{n+1}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}})} \\ \leq\; & e^{C\widetilde{V}^n(T)}\left(\|E_0\|_{B^{\frac{N}{2}}} + \int_0^T e^{-C\widetilde{V}^n(t)}\|\nabla u^n E^n + \nabla\bar{v}E^n + \nabla u^n + \nabla\bar{v}\|_{B^{\frac{N}{2}}}dt\right) \\ \leq\; & e^{C\lambda_1}\left(\|E_0\|_{B^{\frac{N}{2}}} + \|(\nabla u^n,\nabla\bar{v})\|_{L^1_T(B^{\frac{N}{2}})}\|E^n\|_{L^\infty_T(B^{\frac{N}{2}})} + \|(\nabla u^n,\nabla\bar{v})\|_{L^1_T(B^{\frac{N}{2}})}\right) \\ \leq\; & e^{C\lambda_1}\|E_0\|_{B^{\frac{N}{2}}} + 2\lambda_1 e^{C\lambda_1}\|E^n\|_{\widetilde{L}^\infty_T(\widetilde{B}^{\frac{N}{2},\infty}_\mu)} + 2\lambda_1 e^{C\lambda_1}, \end{aligned} \tag{5.6}$$

where $\widetilde{V}^n(t) = \int_0^t \|\nabla u^n\|_{B^{\frac{N}{2}}} + \|\nabla\bar{v}\|_{B^{\frac{N}{2}}}d\tau$. When λ_1 satisfies

$$2\lambda_1 e^{C\lambda_1} \leq \frac{1}{2}, \; e^{C\lambda_1} \leq 2, \; 2\lambda_1 e^{C\lambda_1} \leq \|E_0\|_{B^{\frac{N}{2}}}, \tag{5.7}$$

we have

$$\|E^{n+1}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}})} \le 6\|E_0\|_{B^{\frac{N}{2}}}.$$

From Propositions 3.1 and 4.2, we have

$$\begin{aligned}
&\|u^{n+1}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1})} + \mu\|u^{n+1}\|_{L^1_T(B^{\frac{N}{2}+1})} + \|\nabla p^{n+1}\|_{L^1_T(B^{\frac{N}{2}+1})}\\
\le\ & Ce^{C\widetilde{V}^n(t)}\| -(u^n+\bar v)\cdot\nabla \bar v_i + E^{n+1}_{jk}\partial_j E^{n+1}_{ik} + \partial_j E^{n+1}_{ij}\|_{L^1_T(B^{\frac{N}{2}-1})}\\
\le\ & Ce^{C\lambda_1}\Big(\|(u^n+\bar v)\|_{L^\infty_T(B^{\frac{N}{2}-1})}\|\nabla \bar v\|_{L^1_T(B^{\frac{N}{2}})}\\
& +T\|E^{n+1}\|^2_{L^\infty_T(B^{\frac{N}{2}})} + T\|E^{n+1}\|_{L^\infty_T(B^{\frac{N}{2}})}\Big)\\
\le\ & Ce^{C\lambda_1}\Big(2\lambda_1^2 + T(6\|E_0\|_{B^{\frac{N}{2}}})^2 + 6T\|E_0\|_{B^{\frac{N}{2}}}\Big)\\
\le\ & \lambda_1, \qquad (5.8)
\end{aligned}$$

where we choose λ_1 and T satisfying

$$Ce^{C\lambda_1}\Big(2\lambda_1^2 + T(6\|E_0\|_{B^{\frac{N}{2}}})^2 + 6T\|E_0\|_{B^{\frac{N}{2}}}\Big) \le \lambda_1. \qquad (5.9)$$

Thus, (P_n) hold for all $n \ge 0$. From (5.5), we can easily obtain that u^n_t, E^n_t are uniformly bounded in $L^1_T(B^{\frac{N}{2}-1})$. Then, letting the limit of $\{u^n, E^n\}$ is (u, E), using the classical compactness arguments, we can obtain that $(u+\bar v, E)$ is the solution of (5.4). Thus, we can prove the existence part of Theorem 1.1.

5.2 Further regularity property

Let $s \in (\frac{N}{2}, \frac{N}{2}+1)$. Under the additional assumption $E_0 \in B^s$ and $v_0 \in B^{s-1}$, we obtain that $\bar v \in C([0,T]; B^{s-1})$ and

$$\|\bar v\|_{L^r_T(B^{s-1+\frac{2}{r}})} \le C\|v_0\|_{B^{s-1}}\ r \in [1,\infty]. \qquad (5.10)$$

Then, we shall prove that the sequence $\{(u^n, E^n)\}_{n\in\mathbb{N}}$ is uniformly bounded in U^s_T.

Applying Propositions 3.1 and 4.1, and (P_n), we get

$$\begin{aligned}
&\|E^{n+1}\|_{\widetilde{L}^\infty_T(B^s)}\\
\le\ & e^{C\widetilde{V}^n(T)}\left(\|E_0\|_{B^s} + \int_0^T e^{-C\widetilde{V}^n(t)}\|\nabla u^n E^n + \nabla \bar v E^n + \nabla u^n + \nabla \bar v\|_{B^s}dt\right)\\
\le\ & e^{C\lambda_1}\Big(\|E_0\|_{B^s} + \|(\nabla u^n, \nabla \bar v)\|_{L^1_T(B^{\frac{N}{2}})}\|E^n\|_{L^\infty_T(B^s)}\\
& +\|(\nabla u^n, \nabla \bar v)\|_{L^1_T(B^s)}\|E^n\|_{L^\infty_T(B^{\frac{N}{2}})} + \|(\nabla u^n, \nabla \bar v)\|_{L^1_T(B^s)}\Big)\\
\le\ & e^{C\lambda_1}\|E_0\|_{B^s} + 2\lambda_1 e^{C\lambda_1}\|E^n\|_{\widetilde{L}^\infty_T(B^s)} + Ce^{C\lambda_1}\|v_0\|_{B^{s-1}}(6\|E_0\|_{B^{\frac{N}{2}}}+1)\\
& +e^{C\lambda_1}\|u^n\|_{L^1_T(B^{s+1})}(6\|E_0\|_{B^{\frac{N}{2}}}+1). \qquad (5.11)
\end{aligned}$$

From Propositions 3.1 and 4.2, we have

$$\begin{aligned}
&\|u^{n+1}\|_{\widetilde{L}^\infty_T(B^{s-1})} + \mu\|u^{n+1}\|_{L^1_T(B^{s+1})}\\
\le\ & Ce^{C\widetilde{V}^n(t)}\| -(u^n+\bar v)\cdot\nabla \bar v_i + E^{n+1}_{jk}\partial_j E^{n+1}_{ik} + \partial_j E^{n+1}_{ij}\|_{L^1_T(B^{s-1})}\\
\le\ & Ce^{C\lambda_1}\Big(\|(u^n+\bar v)\|_{L^2_T(B^{\frac{N}{2}})}\|\nabla \bar v\|_{L^2_T(B^{s-1})}\\
& +T\|E^{n+1}\|_{L^\infty_T(B^{\frac{N}{2}})}\|E^{n+1}\|_{L^\infty_T(B^s)} + T\|E^{n+1}\|_{L^\infty_T(B^s)}\Big)
\end{aligned}$$

$$\leq \quad Ce^{C\lambda_1}\lambda_1\|v_0\|_{B^{s-1}} + TCe^{C\lambda_1}\|E^{n+1}\|_{L^\infty_T(B^s)}(6\|E_0\|_{B^{\frac{N}{2}}}+1). \tag{5.12}$$

From (5.11)–(5.12), we have

$$\begin{aligned}
& \|u^{n+1}\|_{\widetilde{L}^\infty_T(B^{s-1})} + \mu\|u^{n+1}\|_{L^1_T(B^{s+1})} + \frac{\mu\|E^{n+1}\|_{L^\infty_T(B^s)}}{2e^{C\lambda_1}(6\|E_0\|_{B^{\frac{N}{2}}}+1)} \\
\leq \quad & Ce^{C\lambda_1}\lambda_1\|v_0\|_{B^{s-1}} + TCe^{C\lambda_1}\|E^{n+1}\|_{L^\infty_T(B^s)}(6\|E_0\|_{B^{\frac{N}{2}}}+1) \\
& + \frac{\mu\|E_0\|_{B^s}}{2(6\|E_0\|_{B^{\frac{N}{2}}}+1)} + \frac{C\mu}{2}\|v_0\|_{B^{s-1}} + \frac{\lambda_1\mu}{(6\|E_0\|_{B^{\frac{N}{2}}}+1)}\|E^n\|_{\widetilde{L}^\infty_T(B^s)} \\
& + \frac{\mu}{2}\|u^n\|_{L^1_T(B^{s+1})}.
\end{aligned} \tag{5.13}$$

From (5.7), we have $\frac{2\lambda_1}{(6\|E_0\|_{B^{\frac{N}{2}}}+1)} \leq \frac{1}{2e^{C\lambda_1}(6\|E_0\|_{B^{\frac{N}{2}}}+1)}$. When $T_1 \in (0,T]$ satisfies

$$2T_1Ce^{C\lambda_1}(6\|E_0\|_{B^{\frac{N}{2}}}+1) \leq \frac{\mu}{4e^{C\lambda_1}(6\|E_0\|_{B^{\frac{N}{2}}}+1)}, \tag{5.14}$$

we can prove by induction from the inequality (5.13) that

$$\begin{aligned}
& \|u^n\|_{\widetilde{L}^\infty_{T_1}(B^{s-1})} + \frac{\mu}{2}\|u^n\|_{L^1_{T_1}(B^{s+1})} + \frac{\mu\|E^n\|_{L^\infty_{T_1}(B^s)}}{8e^{C\lambda_1}(6\|E_0\|_{B^{\frac{N}{2}}}+1)} \\
\leq \quad & Ce^{C\lambda_1}\lambda_1\|v_0\|_{B^{s-1}} + \frac{\mu\|E_0\|_{B^s}}{2(6\|E_0\|_{B^{\frac{N}{2}}}+1)} + \frac{C\mu}{2}\|v_0\|_{B^{s-1}}, \ \forall\ n.
\end{aligned} \tag{5.15}$$

Now, we conclude that the sequence $\{(u^n,E^n)\}_{n\in\mathbb{N}}$ is uniformly bounded in $U^s_{T_1}$. This clearly enables us to prove that the solution (v,E) built in the previous subsection also belongs to $U^s_{T_1}$.

From $(v,E)\in U^{\frac{N}{2}}_T$, we will prove that $(v,E)\in U^s_T$. Applying Propositions 3.1 and 4.1, we get

$$\begin{aligned}
& \|E\|_{\widetilde{L}^\infty_T(B^s)} \\
\leq \quad & e^{C\widetilde{V}(T)}\left(\|E_0\|_{B^s} + \int_0^T e^{-C\widetilde{V}(t)}\|\nabla vE + \nabla v\|_{B^s}dt\right) \\
\leq \quad & C\big(\|E_0\|_{B^s} + \int_0^T \|\nabla v\|_{B^{\frac{N}{2}}}\|E\|_{B^s}dt + \|\nabla v\|_{L^1_T(B^s)}\|E\|_{L^\infty_T(B^{\frac{N}{2}})} \\
& + \|\nabla v\|_{L^1_T(B^s)}\big) \\
\leq \quad & C\|E_0\|_{B^s} + C\int_0^T \|\nabla v\|_{B^{\frac{N}{2}}}\|E\|_{B^s}dt + C\|v\|_{L^1_T(B^{s+1})}.
\end{aligned} \tag{5.16}$$

From Propositions 3.1 and 4.2, we have

$$\begin{aligned}
& \|v\|_{\widetilde{L}^\infty_T(B^{s-1})} + \mu\|v\|_{L^1_T(B^{s+1})} \\
\leq \quad & Ce^{C\widetilde{V}(t)}\left(\|v_0\|_{B^{s-1}} + \|E_{jk}\partial_jE_{ik} + \partial_jE_{ij}\|_{L^1_T(B^{s-1})}\right) \\
\leq \quad & C\|v_0\|_{B^{s-1}} + C\int_0^T (\|E\|_{B^{\frac{N}{2}}}\|E\|_{B^s} + \|E\|_{B^s})dt \\
\leq \quad & C\|v_0\|_{B^{s-1}} + C\int_0^T \|E\|_{B^s}dt.
\end{aligned} \tag{5.17}$$

From (5.16)–(5.17), we have

$$\begin{aligned}
& \|v\|_{\widetilde{L}^\infty_T(B^{s-1})} + \frac{\mu}{2}\|v\|_{L^1_T(B^{s+1})} + \frac{\mu\|E\|_{L^\infty_T(B^s)}}{2C} \\
\leq \quad & C\|E_0\|_{B^s} + C\|v_0\|_{B^{s-1}} + C\int_0^T (\|\nabla v\|_{B^{\frac{N}{2}}}+1)\|E\|_{B^s}dt.
\end{aligned}$$

Using Gronwall's inequality, we can obtain (1.8).

5.3 Uniqueness

Let (v^1, E^1, p^1) and (v^2, E^2, p^2) are solutions of (1.5) satisfying $E^i \in C([0,T]; B^{\frac{N}{2}})$, $v^i \in C([0,T]; B^{\frac{N}{2}-1}) \cap L^1(0,T; B^{\frac{N}{2}})$. From Propositions 3.1 and 4.1, we get

$$\begin{aligned}
&\|E^i - E_0\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1})} \\
\le\ & e^{C\widetilde{V}^i(T)} \left(\int_0^T \|\nabla v^i E^i - v^i \cdot \nabla E_0 + \nabla v^i\|_{B^{\frac{N}{2}-1}} dt \right) \\
\le\ & C\big(\|\nabla v^i\|_{L^1(B^{\frac{N}{2}-1})} \|E^i\|_{L^\infty_T(B^{\frac{N}{2}})} + \|v^i\|_{L^1(B^{\frac{N}{2}})} \|\nabla E_0\|_{L^\infty_T(B^{\frac{N}{2}-1})} \\
& + \|v^i\|_{L^1(B^{\frac{N}{2}})}\big) \\
\le\ & CT^{\frac{1}{2}}.
\end{aligned} \tag{5.18}$$

Let $\tilde{v}$ be the solution of the linear heat equations,

$$\begin{cases} \partial_t \tilde{v}_l - \mu \Delta \tilde{v}_l = E_{0jk}\partial_j E_{0lk} + \partial_j E_{0lj}, \\ \tilde{v}(0,x) = 0. \end{cases} \tag{5.19}$$

It is easy to obtain that $\tilde{v} \in C([0,T]; B^{\frac{N}{2}-1})$ and

$$\|\tilde{v}\|_{L^r_T(B^{\frac{N}{2}-1+\frac{2}{r}})} \le C\|E_{0jk}\partial_j E_{0lk} + \partial_j E_{0lj}\|_{L^1_T(B^{\frac{N}{2}-1})} \le CT,\ r \in [1,\infty]. \tag{5.20}$$

When $N \ge 3$, from Propositions 3.1 and 4.2, we have

$$\begin{aligned}
&\|v^i - \bar{v} - \tilde{v}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-2})} + \mu\|v^i - \bar{v} - \tilde{v}\|_{L^1_T(B^{\frac{N}{2}})} \\
\le\ & C\| - v^i \cdot \nabla(\bar{v} + \tilde{v})_l + E^i_{jk}\partial_j E^i_{lk} - E_{0jk}\partial_j E_{0lk} + \partial_j(E^i_{lj} - E_{0lj})\|_{L^1_T(B^{\frac{N}{2}-2})} \\
\le\ & C\Big(\|v^i\|_{L^\infty(B^{\frac{N}{2}-1})} \|\nabla(\bar{v} + \tilde{v})\|_{L^1(B^{\frac{N}{2}-1})} \\
& + T\|(E^i, E_0)\|_{L^\infty(B^{\frac{N}{2}})} \|E^i - E_0\|_{L^\infty_T(B^{\frac{N}{2}-1})} + T\|E^i - E_0\|_{L^\infty_T(B^{\frac{N}{2}-1})}\Big) \\
\le\ & C(T).
\end{aligned} \tag{5.21}$$

Let $\delta v = v^1 - v^2$ and $\delta E = E^1 - E^2$. Then, we have that

$$(\delta v, \delta E) \in C([0,T]; (B^{\frac{N}{2}-2})^N \times (B^{\frac{N}{2}-1})^{N\times N}).$$

From (1.5), we have

$$\begin{cases}
\nabla \cdot \delta v = \nabla \cdot \delta E^\top = 0, \\
\delta v_{it} + v^1 \cdot \nabla \delta v_i + \delta v \cdot \nabla v^2_i + \partial_i \delta p \\
\quad = \mu \Delta \delta v_i + E^1_{jk}\partial_j \delta E_{ik} + \delta E_{jk}\partial_j E^2_{ik} + \partial_j \delta E_{ij}, \\
\delta E_t + v^1 \cdot \nabla \delta E + \delta v \cdot \nabla E^2 = \nabla v^1 \delta E + \nabla \delta v E^2 + \nabla \delta v, \\
(\delta v, \delta E)(0,x) = (0,0).
\end{cases} \tag{5.22}$$

From Propositions 3.1 and 4.1, we get

$$\begin{aligned}
&\|\delta E\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1})} \\
\le\ & e^{C\widetilde{V}^1(T)} \int_0^T \| - \delta v \cdot \nabla E^2 + \nabla v^1 \delta E + \nabla \delta v E^2 + \nabla \delta v\|_{B^{\frac{N}{2}-1}} dt \\
\le\ & C \int_0^T (\|\delta v\|_{B^{\frac{N}{2}}} \|E^2\|_{B^{\frac{N}{2}}} + \|\nabla v^1\|_{B^{\frac{N}{2}}} \|\delta E\|_{B^{\frac{N}{2}-1}} + \|\delta v\|_{B^{\frac{N}{2}}}) dt \\
\le\ & C\|\delta v\|_{L^1_T(B^{\frac{N}{2}})} + C\int_0^T \|\nabla v^1\|_{B^{\frac{N}{2}}} \|\delta E\|_{B^{\frac{N}{2}-1}} dt.
\end{aligned} \tag{5.23}$$

From Propositions 3.1 and 4.2, we have

$$\begin{aligned}
&\|\delta v\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-2})}+\mu\|\delta v\|_{L^1_T(B^{\frac{N}{2}})} \\
\leq\ & Ce^{C\widetilde{V}^1(T)}\|-\delta v\cdot\nabla v_i^2+E^1_{jk}\partial_j\delta E_{ik}+\delta E_{jk}\partial_j E^2_{ik}+\partial_j\delta E_{ij}\|_{L^1_T(B^{\frac{N}{2}-2})} \\
\leq\ & C\int_0^T\Big[\|\delta v\|_{B^{\frac{N}{2}-2}}\|\nabla v^2\|_{B^{\frac{N}{2}}}+(\|E^1\|_{B^{\frac{N}{2}}}+\|E^2\|_{B^{\frac{N}{2}}}+1)\|\delta E\|_{B^{\frac{N}{2}-1}}\Big]\,dt.
\end{aligned}\tag{5.24}$$

From (5.23)–(5.24), we have

$$\begin{aligned}
&\|\delta E\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1})}+\|\delta v\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-2})}+\|\delta v\|_{L^1_T(B^{\frac{N}{2}})} \\
\leq\ & C\int_0^T\Big[\|\delta v\|_{B^{\frac{N}{2}-2}}\|\nabla v^2\|_{B^{\frac{N}{2}}}+\|\nabla v^1\|_{B^{\frac{N}{2}}}\|\delta E\|_{B^{\frac{N}{2}-1}}+\|\delta E\|_{B^{\frac{N}{2}-1}}\Big]\,dt.
\end{aligned}\tag{5.25}$$

Using Gronwall's inequality, we obtain that $\delta E=\delta v=0$ and finish the proof of uniqueness part of Theorem 1.1 when $N\geq 3$.

When $N=2$, using the similar arguments as that in the proof of (5.18)–(5.21), we have that $(\delta v,\delta E)\in C([0,T];(\dot{B}^{-1}_{2,\infty})^N\times(\dot{B}^0_{2,\infty})^{N\times N})$. From Propositions 3.1 and 4.1, we get

$$\begin{aligned}
&\|\delta E\|_{\widetilde{L}^\infty_t(\dot{B}^0_{2,\infty})} \\
\leq\ & e^{C\widetilde{V}^1(t)}\int_0^t\|-\delta v\cdot\nabla E^2+\nabla v^1\delta E+\nabla\delta v E^2+\nabla\delta v\|_{\dot{B}^0_{2,\infty}}dt \\
\leq\ & C\int_0^t(\|\delta v\|_{B^1}\|\nabla E^2\|_{\dot{B}^0_{2,\infty}}+\|\nabla v^1\|_{B^1}\|\delta E\|_{\dot{B}^0_{2,\infty}}+\|\delta v\|_{\dot{B}^1_{2,\infty}})dt \\
\leq\ & C\|\delta v\|_{L^1_t(B^1)}+C\int_0^t\|\nabla v^1\|_{B^1}\|\delta E\|_{\dot{B}^0_{2,\infty}}ds,\ \ t\in[0,T].
\end{aligned}$$

Now, inserting the following logarithmic inequality (see (4.6) in [10]) in the above estimate,

$$\|f\|_{L^1_T(\dot{B}^1_{2,1})}\lesssim\|f\|_{\widetilde{L}^1_T(\dot{B}^1_{2,\infty})}\ln\Big(e+\frac{\|f\|_{\widetilde{L}^1_T(\dot{B}^0_{2,\infty})}+\|f\|_{\widetilde{L}^1_T(\dot{B}^2_{2,\infty})}}{\|f\|_{\widetilde{L}^1_T(\dot{B}^1_{2,\infty})}}\Big),\tag{5.26}$$

we have for $t\in[0,T]$,

$$\begin{aligned}
\|\delta E\|_{\widetilde{L}^\infty_t(\dot{B}^0_{2,\infty})}\ \leq\ & C\int_0^t\|\nabla v^1\|_{B^1}\|\delta E\|_{\dot{B}^0_{2,\infty}}ds \\
&+C\|\delta v\|_{L^1_t(\dot{B}^1_{2,\infty})}\ln\Big(e+\frac{\|\delta v\|_{\widetilde{L}^1_t(\dot{B}^0_{2,\infty})}+\|\delta v\|_{\widetilde{L}^1_t(\dot{B}^2_{2,\infty})}}{\|\delta v\|_{\widetilde{L}^1_t(\dot{B}^1_{2,\infty})}}\Big).
\end{aligned}$$

Since

$$\|\delta v\|_{\widetilde{L}^1_t(\dot{B}^0_{2,\infty})}+\|\delta v\|_{\widetilde{L}^1_t(\dot{B}^2_{2,\infty})}\leq\sum_{i=1}^2(\|v^i\|_{\widetilde{L}^1_t(\dot{B}^0_{2,1})}+\|v^i\|_{\widetilde{L}^1_t(\dot{B}^2_{2,1})})\leq C_T,$$

we have for $t\in[0,T]$,

$$\begin{aligned}
&\|\delta E\|_{\widetilde{L}^\infty_t(\dot{B}^0_{2,\infty})} \\
\leq\ & C\int_0^t\|\nabla v^1\|_{B^1}\|\delta E\|_{\dot{B}^0_{2,\infty}}ds+C\|\delta v\|_{L^1_t(\dot{B}^1_{2,\infty})}\ln\Big(e+\frac{C_T}{\|\delta v\|_{\widetilde{L}^1_t(\dot{B}^1_{2,\infty})}}\Big),
\end{aligned}$$

and using Gronwall's inequality,

$$\|\delta E\|_{\widetilde{L}^\infty_t(\dot{B}^0_{2,\infty})}\leq C\|\delta v\|_{L^1_t(\dot{B}^1_{2,\infty})}\ln\Big(e+\frac{C_T}{\|\delta v\|_{\widetilde{L}^1_t(\dot{B}^1_{2,\infty})}}\Big).\tag{5.27}$$

From Propositions 3.1 and 4.2, we have

$$
\begin{aligned}
& \|\delta v\|_{\widetilde{L}_t^\infty(\dot{B}_{2,\infty}^{-1})}+\mu\|\delta v\|_{L_t^1(\dot{B}_{2,\infty}^1)} \\
\leq\ & Ce^{C\widetilde{V}^1(t)}\|-\delta v\cdot\nabla v_i^2+E_{jk}^1\partial_j\delta E_{ik}+\delta E_{jk}\partial_j E_{ik}^2+\partial_j\delta E_{ij}\|_{L_t^1(\dot{B}_{2,\infty}^{-1})} \\
\leq\ & C\int_0^t\left[\|\delta v\|_{\dot{B}_{2,\infty}^{-1}}\|\nabla v^2\|_{B^1}+(\|E^1\|_{B^1}+\|E^2\|_{B^1}+1)\|\delta E\|_{\dot{B}_{2,\infty}^0}\right]ds \\
\leq\ & C\int_0^t\left[\|\delta v\|_{\dot{B}_{2,\infty}^{-1}}\|\nabla v^2\|_{B^1}+\|\delta E\|_{L_t^\infty(\dot{B}_{2,\infty}^0)}\right]ds. \qquad (5.28)
\end{aligned}
$$

Denote

$$W(t):=\|\delta v\|_{\widetilde{L}_T^\infty(\dot{B}_{2,\infty}^{-1})}+\mu\|\delta v\|_{L_T^1(\dot{B}_{2,\infty}^1)}.$$

From (5.27)–(5.28), we have

$$W(t)\leq C\int_0^t(1+\|\nabla v^2\|_{B^1})W(s)\ln(e+\frac{C_T}{W(s)})ds,\ T\in[0,t].$$

As

$$\int_0^1\frac{dr}{r\ln(e+\frac{C_T}{r})}=\infty,$$

a slight generalization of Gronwall lemma (see e.g. lemma 3.1 in [4]) implies that $W\equiv 0$ on $[0,T]$. Thus we obtain that $\delta E=\delta v=0$, and finish the proof of uniqueness part of Theorem 1.1 when $N=2$. □

6 A global existence and uniqueness result

This section is devoted to the proof of the following theorem.

Theorem 6.1. *Consider the viscoelastic model (1.5). Suppose that the initial data satisfies the incompressible constraints (1.2)–(1.3),* $E_0\in B^{\frac{N}{2}}\cap B^{\frac{N}{2}-1}$*,* $v_0\in B^{\frac{N}{2}-1}$ *and*

$$\|E_0\|_{B^{\frac{N}{2}}\cap B^{\frac{N}{2}-1}}+\|v_0\|_{B^{\frac{N}{2}-1}}\leq\lambda, \qquad (6.1)$$

where λ *is a small positive constant. Then there exists a global solution for system (1.5) that satisfies*

$$\|(v,E)\|_{X^{\frac{N}{2}}}\leq M(\|E_0\|_{B^{\frac{N}{2}}\cap B^{\frac{N}{2}-1}}+\|v_0\|_{B^{\frac{N}{2}-1}}), \qquad (6.2)$$

where

$$
\begin{aligned}
X^s=\big\{(v,c)\in\ & \big(L^1(\mathbb{R}^+;B^{s+1})\cap C(\mathbb{R}^+;B^{s-1})\big)^N \\
& \times\Big(L^1(\mathbb{R}^+;\tilde{B}_\mu^{s,1})\cap C(\mathbb{R}^+;\tilde{B}_\mu^{s,\infty})\Big)^N\ \big\}.
\end{aligned}
$$

Denote

$$\alpha=\|E_0\|_{\tilde{B}_\mu^{\frac{N}{2},\infty}}+\|v_0\|_{B^{\frac{N}{2}-1}}.$$

From Theorem 1.1, we have that there exists a unique local solution (v,E) of (1.5) with the initial data (v_0,E_0). Assume that the maximum existence time is T^*, such that the solution (v,E) exists on $[0,T^*)$ and

$$(v,E)\in\Big(L^1([0,T^*);B^{\frac{N}{2}+1})\cap C([0,T^*);B^{\frac{N}{2}-1})\Big)^N\times\Big(C([0,T^*);B^{\frac{N}{2}})\Big)^{N\times N}.$$

Using Proposition 3.1, we can easily obtain that $E\in\Big(C([0,T^*);B^{\frac{N}{2}-1})\Big)^{N\times N}$ and omit the details. From Lemmas 2.2–2.3, we have

$$\partial_m E_{ij}-\partial_j E_{im}=E_{lj}\partial_l E_{im}-E_{lm}\partial_l E_{ij}=\partial_l(E_{lj}E_{im}-E_{lm}E_{ij}). \qquad (6.3)$$

We are going to prove the existence of a positive M such that, if α is small enough, the following bound holds,

$$\|(v,E)\|_{X_{T^*}^{\frac{N}{2}}} \le M\alpha. \tag{6.4}$$

Claim 1. If

$$\|(v,E)\|_{X_T^{\frac{N}{2}}} \le 2M\alpha,\ T\in(0,T^*), \tag{6.5}$$

then, we have

$$\|(v,E)\|_{X_T^{\frac{N}{2}}} \le M\alpha. \tag{6.6}$$

when α is small enough.

Let $c=\Lambda^{-1}\nabla\cdot E$, we have

$$\begin{cases} \nabla\cdot v=\nabla\cdot c=0,\\ v_t+v\cdot\nabla v+\nabla p-\mu\Delta v-\Lambda c=G,\\ c_t+v\cdot\nabla c+\Lambda v=L,\\ \Delta E_{ij}=\Lambda\partial_j c_i+\partial_k(\partial_k E_{ij}-\partial_j E_{ik}),\\ (v,c)(0,x)=(v_0,c_0)(x), \end{cases} \tag{6.7}$$

with

$$L_i=\Lambda^{-1}\partial_j(\nabla vE)_{ij}-[\Lambda^{-1}\partial_j,v\cdot]\nabla E_{ij},\ G_i=E_{jk}\partial_j E_{ik}.$$

From Proposition 4.3, we have

$$\begin{aligned} \|(v,c)\|_{X_T^{\frac{N}{2}}} \le\ & Ce^{C\|v\|_{L^1_T(B^{\frac{N}{2}+1})}}\Big(\|c_0\|_{\tilde{B}_\mu^{\frac{N}{2},\infty}}+\|v_0\|_{B^{\frac{N}{2}-1}}\\ &+\|L(s)\|_{L^1_T(\tilde{B}_\mu^{\frac{N}{2},\infty})}+\|G(s)\|_{L^1_T(\tilde{B}^{\frac{N}{2}-1})}\Big). \end{aligned} \tag{6.8}$$

From Proposition 3.2, Lemma 3.1 and (6.5), we obtain

$$\begin{aligned} \|L\|_{L^1_T(\tilde{B}_\mu^{\frac{N}{2},\infty})} &\le \|\Lambda^{-1}\partial_j(\nabla vE)_{ij}\|_{L^1_T(\tilde{B}_\mu^{\frac{N}{2},\infty})}+\|[\Lambda^{-1}\partial_j,v\cdot]\nabla E_{ij}\|_{L^1_T(\tilde{B}_\mu^{\frac{N}{2},\infty})}\\ &\le C\|\nabla vE\|_{L^1_T(\tilde{B}_\mu^{\frac{N}{2},\infty})}+C\|E\|_{L^\infty_T(\tilde{B}_\mu^{\frac{N}{2},\infty})}\|\nabla v\|_{L^1_T(B^{\frac{N}{2}})}\\ &\le C\|E\|_{L^\infty_T(\tilde{B}_\mu^{\frac{N}{2},\infty})}\|\nabla v\|_{L^1_T(B^{\frac{N}{2}})}\le CM^2\alpha^2. \end{aligned} \tag{6.9}$$

From Proposition 3.1 and (6.5), we get

$$\begin{aligned} &\|G(s)\|_{L^1_T(\tilde{B}^{\frac{N}{2}-1})}\le C\|E\|_{L^2_T(B^{\frac{N}{2}})}\|\nabla E\|_{L^2_T(B^{\frac{N}{2}-1})}\le C\|E\|^2_{L^2_T(B^{\frac{N}{2}})}\\ =\ & C\int_0^T\Bigg(\sum_{q\in\mathbb{Z}}\Big(2^{q\frac{N}{2}}\|\Delta_q E\|_{L^2}\max\{\mu,2^{-q}\}\Big)^{\frac12}\\ &\times\Big(2^{q\frac{N}{2}}\|\Delta_q E\|_{L^2}\min\{\mu^{-1},2^q\}\Big)^{\frac12}\Bigg)^2dt\\ \le\ & C\|E\|_{L^\infty_T(\tilde{B}_\mu^{\frac{N}{2},\infty})}\|E\|_{L^1_T(\tilde{B}_\mu^{\frac{N}{2},1})}\le CM^2\alpha^2. \end{aligned} \tag{6.10}$$

From (6.8)–(6.10), we have

$$\|(v,c)\|_{X_T^{\frac{N}{2}}} \le Ce^{CM\alpha}\Big(\|E_0\|_{\tilde{B}_\mu^{\frac{N}{2},\infty}}+\|v_0\|_{B^{\frac{N}{2}-1}}+CM^2\alpha^2\Big). \tag{6.11}$$

From (6.3) and $(6.7)_4$, we have

$$\|E\|_{L^\infty_T(\tilde{B}_\mu^{\frac{N}{2},\infty})} \le C\|c\|_{L^\infty_T(\tilde{B}_\mu^{\frac{N}{2},\infty})}+C\|E^2\|_{L^\infty_T(\tilde{B}_\mu^{\frac{N}{2},\infty})}$$

$$
\begin{aligned}
&\le C\|c\|_{L^\infty_T(\widetilde{B}^{\frac{N}{2},\infty}_\mu)} + C\|E\|_{L^\infty_T(B^{\frac{N}{2}})}\|E\|_{L^\infty_T(\widetilde{B}^{\frac{N}{2},\infty}_\mu)} \\
&\le C\|c\|_{L^\infty_T(\widetilde{B}^{\frac{N}{2},\infty}_\mu)} + CM^2\alpha^2 \qquad (6.12)
\end{aligned}
$$

and

$$
\begin{aligned}
\|E\|_{L^1_T(\widetilde{B}^{\frac{N}{2},1}_\mu)} &\le C\|c\|_{L^1_T(\widetilde{B}^{\frac{N}{2},1}_\mu)} + C\|E^2\|_{L^1_T(\widetilde{B}^{\frac{N}{2},1}_\mu)} \\
&\le C\|c\|_{L^1_T(\widetilde{B}^{\frac{N}{2},1}_\mu)} + C\|E^2\|_{L^1_T(B^{\frac{N}{2}})} \\
&\le C\|c\|_{L^1_T(\widetilde{B}^{\frac{N}{2},1}_\mu)} + C\|E\|^2_{L^2_T(B^{\frac{N}{2}})} \\
&\le C\|c\|_{L^1_T(\widetilde{B}^{\frac{N}{2},1}_\mu)} + CM^2\alpha^2. \qquad (6.13)
\end{aligned}
$$

From (6.11)–(6.13), we have

$$
\begin{aligned}
\|(v,E)\|_{X^{\frac{N}{2}}_T} &\le Ce^{CM\alpha}\left(\|E_0\|_{\tilde{B}^{\frac{N}{2},\infty}_\mu} + \|v_0\|_{B^{\frac{N}{2}-1}} + CM^2\alpha^2\right) + CM^2\alpha^2 \\
&\le 2C\alpha + \alpha \le M\alpha. \qquad (6.14)
\end{aligned}
$$

when $M = 2C+1$ and α satisfies

$$
e^{CM\alpha} \le 2,\ 2C^2M^2\alpha + CM^2\alpha \le 1. \qquad (6.15)
$$

Then, we finish the proof of Claim 1. From the classical continuation method and Claim 1, we can easily obtain that (6.4). Combining Theorem 1.1, one can obtain that $T^* = \infty$ and finish the proof of Theorem 1.2. □

References

[1] R. Agemi, *Global existence of nonlinear elastic waves*, Invent. Math. , 142 (2000), pp. 225–250.

[2] R.Byron Bird, C.F. Curtiss, R.C. Armstrong and O. Hassager, *Dynamics of polymeric liquids, Kinetic Theory*, vol. 2, 2nd Edition. Wiley Interscience, New York, (1987).

[3] M. Cannone, *Ondelettes, paraproduits et Navier–Stokes*, Nouveaux essais, Diderot Editeur, Paris, (1995).

[4] J.-Y. Chemin and N. Lerner, *Flot de champs de vecteurs non lipschitziens et équations de Navier–Stokes*, J. Diff. Equations, 121 (1992), pp. 314–328.

[5] J.Y. Chemin, *Théorèmes d'unicité pour le système de Navier–Stokes tridimensionnel*, J. Anal. Math., 77 (1999), pp. 27–50.

[6] J.Y. Chemin and I. Gallagher, *Wellposedness and stability results for the Navier–Stokes equations in* $\mathbb{R}^3$, Ann. Inst. H. Poincaré Anal. Non Linéaire , 26 (2009), pp. 599–624.

[7] C. Dafermos, *Hyperbolic conservation laws in continuum physics*, Springer-Verlag, Berlin, (2000).

[8] R. Danchin, *Global existence in critical spaces for compressible Navier–Stokes equations*, Invent. Math., 141 (2000), pp. 579–614.

[9] R. Danchin, *A few remarks on the Camassa-Holm equation*, Differential Integral Equations, 14 (2001), pp. 953–988.

[10] R. Danchin, *Density-dependent incompressible viscous fluids in critical spaces*, Proc. Roy. Soc. Edinburgh Sect. A, 133 (2003), pp. 1311–1334.

[11] R. Danchin, *On the uniqueness in critical spaces for compressible Navier–Stokes equations*, NoDEA Nonlinear Differential Equations Appl., 12 (2005), pp. 111–128.

[12] H. Fujita and T. Kato, *On the Navier–Stokes initial value problem I*, Arch. Rational Mech. Anal., 16 (1964), pp. 269–315.

[13] M.E. Gurtin, *An introduction to continuum mechanics*, Academic Press, New York-London, (1981).

[14] S. Kawashima and Y. Shibata, *Global existence and exponential stability of small solutions to nonlinear viscoelasticity*, Comm. Math. Phys., 148 (1992), pp. 189–208.

[15] H. Koch and D. Tataru, *Well-posedness for the Navier–Stokes equations*, Adv. Math., 157 (2001), pp. 22–35.

[16] R.G. Larson, *The Structure and Rheology of Complex Fluids*, Oxford University Press, New York, (1995).

[17] Z. Lei, C. Liu and Y. Zhou, *Global solutions for incompressible viscoelastic fluids*, Arch. Ration. Mech. Anal., 188 (2008), pp. 371–398.

[18] Z. Lei, C. Liu and Y. Zhou, *Global existence for a 2D incompressible viscoelastic model with small strain*, Commun. Math. Sci., 5 (2007), pp. 595–616.

[19] Z. Lei and Y. Zhou, *Global existence of classical solutions for the two–dimensional Oldroyd model via the incompressible limit*, SIAM J. Math. Anal., 37 (2005), pp. 797–814.

[20] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math., 63 (1934), pp. 193–248.

[21] F.H. Lin and C. Liu, *Nonparabolic dissipative systems modeling the flow of liquid crystals*, Comm. Pure Appl. Math., 48 (1995), pp. 501–537.

[22] F.H. Lin, C. Liu and P. Zhang, *On hydrodynamics of viscoelastic fluids*, Comm. Pure Appl. Math., 58 (2005), pp. 1437–1471.

[23] J.L. Lions, *On some questions in boundary value problems of mathematical physics, Contemporary developments in continuum mechanics and partial differential equations*, North-Holland, Amsterdam-New York, (1978)

[24] C. Liu and N.J. Walkington, *An Eulerian description of fluids containing visco-elastic particles*, Arch. Ration. Mech. Anal., 159 (2001), pp. 229–252.

[25] Qian, J.Z.: Well-posedness in critical spaces for incompressible viscoelastic fluid system. Nonlinear Anal. **72**, 3222-3234, (2010)

[26] T.C. Sideris, *Nonresonance and global existence of prestressed nonlinear elastic waves*, Ann. of Math., 151 (2000), pp. 849–874.

[27] T.C. Sideris and B. Thomases, *Global existence for three–dimensional incompressible isotropic elastodynamics via the incompressible limit*, Comm. Pure Appl. Math., 58 (2005), pp. 750–788.

[28] T.C. Sideris and B. Thomases, *Global existence for three–dimensional incompressible isotropic elastodynamics*, Comm. Pure Appl. Math., 60 (2007), pp. 1707–1730.